# PUROHIT

**Type 1**:

a. $\int \sin^n x \ . \ \cos^m x \ dx$

if n or m is odd split one odd power. If both n and m are odd then split one power of smaller odd number.

b. $\int \frac{\sin^n x}{\cos^m x} dx$ OR $\int \frac{\cos^n x}{\sin^m x} dx$

if n is odd split one odd power of n

1. $\frac{\cos^9 x}{\sin x}$
2. $\sin^5 x$
3. $\sin^3 x \ \cos^3 x$
4. $\sin^4 x \ \cos^3 x$
5. $\sin^3 x \cos^{10} x$
6. $\sin^5 x \cos x$
7. $\cos^5 x \ \sin x$
8. $\cos^7$ x

***Answers***:

1. $\log \sin x - 2\sin^2 x + \frac{3}{2}\sin^4 x - \frac{2}{3}\sin^6 x + \frac{1}{8}\sin^8 x + c$
2. $-\cos x - \frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} + c$
3. $\frac{1}{4} \ \sin^4 x - \frac{1}{6}\sin^6 x + c$
4. $\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$
5. $\frac{-\cos^{11} x}{11} + \frac{\cos^{13} x}{13} + c$
6. $\frac{\sin^6 x}{6}$
7. $\frac{-\cos^6 x}{6}$
8. $\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

**Type 2**:

$\int \frac{1}{\cos^m x \sin^n x} dx$ where $m+n$ is even

Multiply by $\sec^{m+n} x$ in Nr and Dr we have $\int \frac{\sec^{m+n} x}{\tan^n x} dx$ being $m+n$ is even we split $\sec^2 x$ and use identity and put $\tan x = t$

$$\int \frac{\sec^{m+n} x}{\tan^n x} dx = \int \frac{\sec^{m+n-2} x}{\tan^n x} \sec^2 x dx = \int \frac{\left(1+\tan^2\right)^{\frac{m+n-2}{2}}}{\tan^n x} \sec^2 x dx$$

9. $\dfrac{1}{\cos^2 x \sin^4 x}$ 10. $\dfrac{1}{\cos^5 x \sin^3 x}$ 11. $\dfrac{1}{\cos x \sin^3 x}$ 12. $\dfrac{1}{\cos^3 x \sin x}$

13. $\sec^{4/3} x \operatorname{cosec}^{8/3} x$ 14. $\sqrt[3]{\dfrac{\sin^2 x}{\cos^{14} x}}$ 15. $\dfrac{1}{\sqrt{\sin^3 x \cos^5 x}}$ 16. $\dfrac{\sin^4 x}{\cos^8 x}$

***Answers***:

9. $-\dfrac{1}{3}\cot^3 x - 2\cot x + \tan x + c$ 10. $-\dfrac{1}{2}\dfrac{1}{\tan^2 x} + 3\log\tan x + \dfrac{3}{2}\tan^2 x + \dfrac{1}{4}\tan^4 x + c$

11. $-\dfrac{1}{2}\dfrac{1}{\tan^2 x} + \log\tan x + c$ 12. $\dfrac{1}{2}\tan^2 x + \log\tan x + c$ 13. $\dfrac{-3}{5}\tan^{-5/3} x + 3\tan^{1/3} x + c$

14. $\dfrac{3}{5}\tan^{5/3} x + \dfrac{3}{11}\tan^{11/3} x + c$ 15. $\dfrac{-2}{\sqrt{\tan x}} + \dfrac{2}{3}\tan^{3/2} x + c$ 16. $\dfrac{\tan^5 x}{5} + \dfrac{\tan^7 x}{7} + c$

**Type 3:**

a. $\int \tan^{odd} x \sec^{any} x\, dx$ split one power of $\tan x$ and $\sec x$ each. And put $\sec x = t$ And use fundamental identity. <u>This can be done by using type 1 also</u>

$$\int \tan^5 x \sec^{\frac{3}{2}} x\, dx = \int \tan^4 x \sec^{\frac{1}{2}} x \ \tan x \sec x dx = \int \left(\sec^2 x - 1\right)^2 x \sec^{\frac{1}{2}} x \ \tan x \sec x dx$$

$$= \int \left(t^2 - 1\right)^2 x\sqrt{t}\ dt$$

b. $\int \tan^{any} x \sec^{even} x\, dx$ split 2 power of $\sec x$. And put $\tan x = t$ And use fundamental identity

$$\int \tan^{\frac{5}{2}} x \sec^4 x\, dx = \int \tan^{\frac{5}{2}} x \sec^2 x \ \sec^2 x dx = \int \tan^{\frac{5}{2}} x\left(1 + \tan^2 x\right)\ \sec^2 x dx$$

$$= \int \mathrm{t}^{\frac{5}{2}}\left(1 + t^2\right) dt$$

c. $\int \cot^{odd} x \operatorname{cosec}^{any} x\, dx$ split one power of $\cot x$ and $\operatorname{cosec} x$ each. And put $\operatorname{cosec} x = t$ And use fundamental identity <u>This can be done by using type 1 also</u>

d. $\int \cot^{any} x \operatorname{cosec}^{even} x\, dx$ split 2 power of $\operatorname{cosec} x$. And put $\cot x = t$ And use fundamental identity.

Note: $\int \tan^{even} x \sec^{odd} x\ dx$ we can't use this method

**Important**: Try to split $\sec^2 x$, whether we are able to use identity if not then we split $\sec x \tan x$ whether we still able to use identity . In this $\int \tan^{even} x \sec^{odd} x\ dx$ we can't use identity by splitting $\sec^2 x$ or $\sec x \tan x$.

In the same way we consider in $\int \cot^{any} x \text{ cosec}^{even} x\ dx$

17. $\sqrt{\tan x}\sec^4 x$ 18. $\tan^3 x \sec^3 x$ 19. $\cot^4 x \text{ cosec}^4 x$ 20. $\cot^5 x \text{ cosec}^{3/2} x$

***Answers*:**

17. $\frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + c$

18. $\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + c$

19. $-\frac{1}{5}\cot^5 x - \frac{1}{7}\cot^7 x + c$

20. $-\frac{2}{7}\text{cosec}^{7/2} x + \frac{2}{3}\text{cosec}^{3/2} x + c$

**Type 4**:

Use of reduction method

a. $\int \tan^n x\ dx$, if n is odd use type 1. If n is even and if $n \geq 4$

$$\int \tan^n x\ dx = \int \tan^{n-2} x \tan^2 \text{x}\ dx = \int \tan^{n-2} x\left(\sec^2 x - 1\right)\ dx = \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x\ dx$$

b. $\int \cot^n x\ dx$ if n is odd use type 1. If n is even and if $n \geq 4$ similarly as above.

21. $\tan^4 x$ 22. $\tan^5 x$ 23.. $\cot^6 x$ 24. $\cot^5 x$

***Answers*:**

21. $\frac{1}{3}\tan^3 x - \tan x + x + c$

22. $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log\sec x + c$

23. $-\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot - x + c$

24. $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log\sin x + c$

**Type 5**:

Use of reduction method (by parts)

a. $\int \sec^n x\, dx$, if n is even use type 3b and if n is odd and if $n \ge 3$ (use by parts)

$$I_n = \int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-3} x \sec x \tan x \tan x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \left(\sec^n x - \sec^{n-2} x\right)\, dx = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

b. $\int \text{cosec}^n x\, dx$ if n is even use type 3d and if n is odd and if $n \ge 3$ similarly as above

25. $\sec^4 2x$ 26. $\text{cosec}^4 3x$ 27. $\sec^3 x$

***Answers:***

25. $\frac{1}{6}\tan^3 2x + \frac{1}{2}\tan 2x + c$

26. $-\frac{1}{9}\cot^3 3x - \frac{1}{3}\cot 3x + c$

27. $\frac{1}{2}\left[\sec x \tan x + \log(\sec x + \tan x)\right]$

**Type 6**:

a. $\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\, dx$ where a b, c, d are any real numbers

$Nr. = A\left(diff\ \ of\ \ Dr.\right) + B\left(Dr.\right)$

Write $a\cos x + b\sin x = A\left(-c\sin x + d\cos x\right) + B\left(c\cos x + d\sin x\right)$ and compare the coefficient of $\sin x\ and\ \cos x$ to find A and B.

Short methods: $\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\, dx = \left(\frac{ac + bd}{c^2 + d^2}\right)x + \left(\frac{ad - bc}{c^2 + d^2}\right)\log\left|c\cos x + d\sin x\right| + c$

28. $\frac{1}{1 - \tan x}$ 29. $\frac{1}{1 + \cot x}$ 30. $\frac{2\sin x + 3\cos x}{5\sin x + 4\cos x}$ 31. $\frac{2\tan x + 3}{3\tan x + 4}$

32. $\frac{1}{3 + 4\cot x}$ 33. $\frac{1}{4 + 3\tan x}$ 34. $\frac{8\cot x + 1}{3\cot x + 2}$ 35. $\frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}$

***Answers*:**

28. $\frac{1}{2}\left[x-\log(\cos x-\sin x)\right]+c$

29. $\frac{1}{2}\left[x-\log(\cos x+\sin x)\right]+c$

30. $\frac{22}{41}x+\frac{7}{41}\log(5\sin x+4\cos x)+c$

31. $\frac{18}{25}x+\frac{1}{25}\log\left(3\sin x+4\cos x\right)+c$

32. $\frac{3}{25}x-\frac{4}{25}\log\left(3\sin x+4\cos x\right)+c$

33. $\frac{4}{25}x+\frac{3}{25}\log\left(3\sin x+4\cos x\right)+c$

34. $2x+\log\left(2\sin x+3\cos x\right)+c$

35. $\frac{12}{13}x-\frac{5}{13}\log\left(3\cos x+2\sin x\right)+c$

**Type 7**:

| | | |
|---|---|---|
| a. | $\int \cos mx.\sin nx\ dx$ | Use $\frac{1}{2}\left[\sin(Sum)\pm\sin(Difference)\right]$ |
| b. | $\int \cos mx.\cos nx\ dx$ | Use $\frac{1}{2}\left[\cos(Sum)+\cos(Difference)\right]$ |
| c. | $\int \sin mx.\sin nx\ dx$ | Use $-\frac{1}{2}\left[\cos(Sum)-\cos(Difference)\right]$ |

36. sin 3x .cos 5x

37. cos x cos 2x cos 3x

38. sin 2x sin 3x sin 5x

39. sin 4x cos 3x

40. $\frac{\sin 4x}{\sin x}$

41..sin 4x cos 7x

***Answers*:**

36. $\frac{1}{2}\left(-\frac{\cos 8x}{8}+\frac{\cos 2x}{2}\right)+c$

37. $\frac{1}{4}\left(x+\frac{\sin 2x}{2}+\frac{\sin 4x}{4}+\frac{\sin 6x}{6}\right)+c$

38. $-\frac{1}{4}\left(\frac{\cos 6x}{6}+\frac{\cos 4x}{4}-\frac{\cos 10x}{10}\right)+c$

39. $\frac{1}{2}\left(-\frac{\cos 7x}{7}-\cos x\right)+c$

40. $2\left(\frac{\sin 3x}{3}+\sin x\right)+c$

41. $-\frac{1}{22}\cos 11x+\frac{1}{6}\cos 3x+c$

**Type 8**:

| | | |
|---|---|---|
| a. | $\int\frac{1}{1\pm\sin x}dx$ , $\int\frac{1}{1\pm\cos x}dx$ | divide and multiply by conjugate |
| b. | $\int\frac{1}{\sqrt{1\pm\cos x}}dx$ , $\int\sqrt{1\pm\cos x}dx$ , $\int\sqrt{1\pm\sin x}dx$ | Use half angle formula |
| c. | $\int\frac{1}{\sqrt{1\pm\sin x}}dx$ | write sin x = $\cos\left(\frac{\pi}{2}-x\right)$ and then use half angle formula |

42. $\frac{1-\cos x}{1+\cos x}$ 43. $\frac{\sec x}{\sec x+\tan x}$ 44. $\frac{co\sec x}{co\sec x-\cot x}$ 45. $\sqrt{1+\cos x}$

46. $\frac{1}{\sqrt{1+\sin x}}$ 47. $\frac{1}{\sqrt{1-\cos x}}$ 48. $\sqrt{1+\sin 3x}$ 49. $\frac{\tan x}{\sec x+\tan x}$

50. $\frac{1}{1+\cos 2x}$ 51. $\frac{\cot x}{\cos ecx-\cot x}$ 52. $\sqrt{1-\cos 3x}$ 53. $\frac{1+\sin x}{1-\sin x}$

***Answers*:**

42. – 2 cot x – x + 2 cosec x + c 43. tan x – sec x + c 44. – cot x – cosec x + c

45. $2\sqrt{2}\sin\left(\frac{x}{2}\right)+\text{c}$ 46. $-\sqrt{2}\log\left[\sec\left(\frac{\pi}{4}-\frac{x}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]+c$

47. $\sqrt{2}\log\left[\cos ec\left(\frac{x}{2}\right)-\cot\left(\frac{x}{2}\right)\right]+c$ 48. $\frac{2}{3}\left(\sin\frac{3x}{2}-\cos\frac{3x}{2}\right)+c$

49. sec x – tan x + x + c 50. $-\frac{\cot 2x}{2}+\frac{\cos ec2x}{2}+c$ 51. – x – cot x – cosec x

52. $-\frac{2\sqrt{2}}{3}\cos\frac{3x}{2}$ 53. 2 tan x + 2 sec x – x

**Type 9**:

a. $\int\frac{\left(x^n\pm x^m\right)^{p/n}}{x^{p+n-m+1}}dx=\int\frac{x^p\left(1\pm\frac{1}{x^{n-m}}\right)^{p/n}}{x^{p+n-m+1}}dx=\int\frac{\left(1\pm\frac{1}{x^{n-m}}\right)^{p/n}}{x^{n-m+1}}dx$ put $\left(1\pm\frac{1}{x^{n-m}}\right)=t$

b. $\int\frac{1}{\sqrt[p]{\left(x\pm a\right)^m\left(x\pm b\right)^{2p-m}}}dx=\int\frac{1}{\sqrt[p]{\frac{\left(x\pm a\right)^m}{\left(x\pm b\right)^m}\left(x\pm b\right)^{2p}}}dx=\int\frac{1}{\sqrt[p]{\left(\frac{x\pm a}{x\pm b}\right)^m}\left(x\pm b\right)^2}dx$ put $\frac{x\pm a}{x\pm b}=t$

54. $\frac{\left(x^4-x\right)^{1/4}}{x^5}$ 55. $\frac{1}{x^2\left(x^4+1\right)^{3/4}}$ 56. $\frac{\sqrt{x^2+1}}{x^4}$ 57. $\frac{1}{\sqrt[4]{\left(x-1\right)^3\left(x+2\right)^5}}$

***Answers*:**

54. $\frac{4}{15}\left(1-\frac{1}{x^3}\right)^{1/4}+c$ 55. $-\left(1+\frac{1}{x^4}\right)^{1/4}+c$ 56. $-\frac{1}{3}\frac{\left(x^2+1\right)^{3/2}}{x^3}+c$

57. $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+c$

**Type 10: (Standard Form)**

| | | | |
|---|---|---|---|
| a. | $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left\|\frac{x-a}{x+a}\right\|+c$ | b. | $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left\|\frac{a+x}{a-x}\right\|+c$ |
| c. | $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$ | d. | $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+\text{c}$ |
| e. | $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left\|x+\sqrt{x^2+a^2}\right\|+\text{c}$ | f. | $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left\|x+\sqrt{x^2-a^2}\right\|+\text{c}$ |
| g. | $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right]+\text{c}$ | | |
| h. | $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2+x^2}+a^2\log\left\|x+\sqrt{x^2+a^2}\right\|\right]+\text{c}$ | | |
| i. | $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2}-a^2\log\left\|x+\sqrt{x^2-a^2}\right\|\right]+\text{c}$ | | |

58. $\frac{1}{\sqrt{9-25x^2}}$
59. $\frac{1}{\sqrt{a^2+b^2x^2}}$
60. $\frac{1}{9x^2-1}$
61. $\frac{1}{3+2x^2}$
62. $\frac{1}{\sqrt{4x^2-7}}$
63. $\frac{1}{7-3x^2}$
64. $\sqrt{5-2x^2}$
65. $\sqrt{6x^2+3}$
66. $\sqrt{3x^2-5}$
67. $\frac{x^2-1}{x^2+4}$
68. $\frac{x^4+1}{x^2+1}$

***Answers***:

58. $\frac{1}{5}\sin^{-1}\frac{5x}{3}+c$
59. $\frac{1}{b}\log\left(x+\sqrt{\frac{a^2}{b^2}+x^2}\right)+c$
60. $\frac{1}{6}\log\left|\frac{3x-1}{3x+1}\right|+c$
61. $\frac{1}{\sqrt{6}}\tan^{-1}\frac{\sqrt{2}x}{\sqrt{3}}$
62. $\frac{1}{2}\log\left(x+\sqrt{x^2-\frac{7}{4}}\right)$
63. $\frac{1}{2\sqrt{21}}\log\left(\frac{\sqrt{7}+\sqrt{3}x}{\sqrt{7}-\sqrt{3}x}\right)$
64. $\frac{1}{2}x\sqrt{5-2x^2}+\frac{5}{2\sqrt{2}}\sin^{-1}\frac{\sqrt{2}x}{\sqrt{5}}$
65. $\frac{\sqrt{6}}{2}\left[x\sqrt{x^2+\frac{1}{2}}+\frac{1}{2}\log\left(x+\sqrt{x^2+\frac{1}{2}}\right)\right]$
66. $\frac{\sqrt{3}}{2}\left[x\sqrt{x^2-\frac{5}{3}}-\frac{5}{3}\log\left(x+\sqrt{x^2-\frac{5}{3}}\right)\right]$
67. $x-\frac{5}{2}\tan^{-1}\frac{x}{2}+c$
68. $\frac{x^3}{3}-x+2\tan^{-1}x+c$

**Type 11: (Perfect Square Method)**

a. $\int \frac{1}{ax^2 + bx + c} dx$ ⇒ converted in to type 10a or 10b or 10c after making perfect square

b. $\int \frac{1}{\sqrt{ax^2 + bx + c}} dx$ ⇒ converted in to type 10d or 10e or 10f after making perfect square

c. $\int \sqrt{ax^2 + bx + c}\ dx$ ⇒ converted in to type 10g or 10h or 10i after making perfect square

How to make perfect square

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Short methods: $\int \frac{1}{ax^2 + bx + c} dx = \frac{2}{\sqrt{-D}} \tan^{-1}\left(\frac{\left(ax^2 + bx + c\right)'}{\sqrt{-D}}\right) + c,\ \ if\ \ D < 0$

69. $\frac{x}{x^4 + x^2 + 1}$ 70. $\frac{1}{\sqrt{x(1-2x)}}$ 71. $\frac{1}{\sqrt{8 + 3x - x^2}}$ 72. $\sqrt{1 - 4x - x^2}$

73. $\sqrt{x^2 + 3x}$ 74. $\sqrt{6 + x - 2x^2}$ 75. $\frac{1}{13x^2 + 13x - 10}$ 76. $\frac{1}{3 + 2x - x^2}$

77. $\frac{1}{\sqrt{2x^2 + 3x - 2}}$ 78. $\frac{1}{\sqrt{3x^2 + 5x + 7}}$ 79. $\sqrt{2x^2 + 3x + 4}$

**Answers:**

69. $\frac{1}{\sqrt{3}} \tan^{-1} \frac{2x^2 + 1}{\sqrt{3}} + c$ 70. $\frac{1}{\sqrt{2}} \sin^{-1}(4x - 1)$ 71. $\sin^{-1} \frac{2x - 3}{\sqrt{41}}$

72. $\frac{1}{2}\left[(x+2)\sqrt{-x^2 - 4x + 1} + 5\sin^{-1}\frac{x+2}{\sqrt{5}}\right]$

73. $\frac{1}{2}\left[\left(x + \frac{3}{2}\right)\sqrt{x^2 + 3x} - \frac{9}{4}\log\left\{\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x}\right\}\right]$

74. $\frac{1}{\sqrt{2}}\left[\left(x - \frac{1}{4}\right)\sqrt{-x^2 + \frac{1}{2}x + 3} + \frac{49}{16}\sin^{-1}\left(\frac{4x-1}{7}\right)\right]$ 75. $\frac{1}{\sqrt{689}}\log\left|\frac{\sqrt{13}(2x+1) - \sqrt{53}}{\sqrt{13}(2x+1) + \sqrt{53}}\right| + c$

76. $\frac{1}{4}\log\left|\frac{x+1}{3-x}\right|+c$

77. $\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x-1}\right|+c$

78. $\frac{1}{\sqrt{3}}\log\left|\left(x+\frac{5}{6}\right)+\sqrt{x^2+\frac{5}{3}x+\frac{7}{3}}\right|+c$

79. $\left(\frac{4x+3}{8}\right)\sqrt{2x^2+3x+4}+\frac{23\sqrt{2}}{32}\log\left|\left(\frac{4x+3}{8}\right)+\sqrt{x^2+\frac{3}{2}x+2}\right|+c$

**Type 12: (Some adjustment)**

a. $\int\frac{px+q}{ax^2+bx+c}\,dx$ $\Rightarrow$ converted in to type 11a after making some adjustment

b. $\int\frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ $\Rightarrow$ converted in to type 11b after making some adjustment

c. $\int(px+q)\sqrt{ax^2+bx+c}\,dx$ $\Rightarrow$ converted in to type 11c after making some adjustment

How to make some adjustment:

px + q = $\frac{p}{2a}\left(2ax+\frac{2aq}{p}+b-b\right)=\frac{p}{2a}\left[\left(2ax+b\right)+\left(\frac{2aq}{p}-b\right)\right]$

80. $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$ 81. $\frac{x}{\sqrt{8+x-x^2}}$ 82. $x\sqrt{1+x-x^2}$ 83. $(2x-5)\sqrt{x^2-4x+3}$

84. $(3x+1)\sqrt{1-4x-x^2}$ 85. $\frac{6x+5}{\sqrt{6+x-2x^2}}$ 86. $\frac{x^3}{x^4+3x^2+2}$

87. $\frac{x^3+x^2+2x+1}{x^2-x+1}$

***Answers*:**

80. $6\sqrt{x^2-9x+20}+34\ \log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]$

81. $-\sqrt{8+x-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)$

82. $-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{1}{4}\left[\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2}+\frac{5}{4}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right]$

83. $\frac{2}{3}\left(x^2-4x+3\right)^{3/2}-\frac{1}{2}\left[(x-2)\sqrt{x^2-4x+3}-\log\left\{(x-2)+\sqrt{x^2-4x+3}\right\}\right]$

84. $-\left(1-4x-x^2\right)^{3/2}-\frac{5}{2}\left[(x+2)\sqrt{1-4x-x^2}+5\sin^{-1}\frac{x+2}{\sqrt{5}}\right]$

85. $-3\sqrt{6+x-2x^2}+\frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)$ 86. $\frac{1}{4}\log\left(x^4+3x^2+2\right)-\frac{3}{4}\log\left(\frac{x^2+1}{x^2+2}\right)+c$

87. $\frac{1}{2}x^2+2x+\frac{3}{2}\log\left(x^2-x+1\right)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+c$

**Type 13:**

a. $\int\frac{px^2+qx+r}{ax^2+bx+c}\,dx$ ⇒ converted in to type 11a after making some adjustment

b. $\int\frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}\,dx$ ⇒ converted in to type 11b after making some adjustment

c. $\int\left(px^2+qx+r\right)\sqrt{ax^2+bx+c}\,dx$ ⇒ converted in to type 12c after making some adjustment

Numerator = A (denominator) + B (differ. Of denominator) + C

$px^2+qx+r=A\left(ax^2+bx+c\right)+B\left(\frac{d}{dx}\left(ax^2+bx+c\right)\right)+C$ comparing the coefficient of like terms, we get the value of A, B and C

88. $\int\frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$

***Answer***: 88 . $\left(x+\frac{7}{2}\right)\sqrt{x^2+x+1}+\frac{5}{4}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+c$

***Solution***: $2x^2+5x+4=A\left(x^2+x+1\right)+B\left(2x+1\right)+C$ comparing coefficient, we get

$\Rightarrow 2x^2+5x+4=2\left(x^2+x+1\right)+\frac{3}{2}\left(2x+1\right)+\frac{1}{2}$

$$\int\frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx=2\int\frac{x^2+x+1}{\sqrt{x^2+x+1}}\,dx+\frac{3}{2}\int\frac{2x+1}{\sqrt{x^2+x+1}}\,dx+\frac{1}{2}\int\frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$=\underbrace{2\int\sqrt{x^2+x+1}\,dx}_{T-11}+\frac{3}{2}\int\frac{dt}{\sqrt{t}}+\frac{1}{2}\underbrace{\int\frac{1}{\sqrt{x^2+x+1}}\,dx}_{T-11}$$

**Type 14**:

a. $\int \frac{1}{a+b\sin^2 x+c\cos^2 x+d\sin x\cos x}\,dx$ Where a, b, c and d are any real numbers

Step I multiply numerator and denominator both by $\sec^2 x$

Step II replace $\sec^2 x$, if any in, in denominator by $1+\tan^2 x$.

Step III Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

89. $\frac{1}{1+3\sin^2 x}$

90. $\frac{1}{1+3\sin^2 x+8\cos^2 x}$

91. $\frac{1}{4-5\sin^2 x}$

92. $\frac{1}{(2\sin x+3\cos x)^2}$

93. $\frac{1}{3+\sin 2x}$

94. $\frac{1}{2-3\cos 2x}$

95. $\frac{\sin x}{\sin 3x}$

96. $\frac{1}{\cos x(\sin x+2\cos x)}$

***<u>Answers</u>*:**

89. $\frac{1}{2}\tan^{-1}(2\tan x)$

90. $\frac{1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right)$

91. $\frac{1}{4}\log\left(\frac{2+\tan x}{2-\tan x}\right)$

92. $-\frac{1}{2(2\tan x+3)}+c$

93. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{3\tan x+1}{2\sqrt{2}}\right)+c$

94. $\frac{1}{2\sqrt{5}}\log\left(\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right)+c$

95. $\frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right)+c$

96. log (tan x + 2) + c

**Type 15**:

a. $\int \frac{1}{a+b\sin x+c\cos x}$ where a, b c are any real numbers

put $\tan\frac{x}{2}=t$, $dx=\frac{2dt}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$, $\sin x=\frac{2t}{1+t^2}$

after performing this substitution, above type is converted into type 11 (perfect square method).

If in above a = 0 we can we substitute $b=r\cos\theta,\ c=r\sin\theta\ and\ so\ r^2=\left(a^2+b^2\right)$,

and put $b\sin x+c\cos x=r\sin(x+\theta)\ or\ r\cos(x-\theta)$

97. $\frac{1}{5+4\cos x}$

98. $\frac{1}{3+2\sin x+\cos x}$

99. $\frac{1}{3\cos x+4\sin x}$

100. $\frac{1}{5-4\sin x}$

101. $\frac{1+\sin x}{\sin x(1+\cos x)}$

***Answers***:

97. $\frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right)$

98. $\tan^{-1}\left(1+\tan\frac{x}{2}\right)$

99. $\frac{1}{5}\log\left(\frac{\frac{1}{3}+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right)$ OR $\frac{1}{5}\log\left(\frac{5+3\sin x-4\cos x}{3\cos x+4\sin x}\right)$

100. $\frac{2}{3}\tan^{-1}\left(\frac{5\tan\frac{x}{2}-4}{3}\right)+c$

101. $\frac{1}{2}\left[\log\left(\tan\frac{x}{2}\right)+\frac{\tan^2\frac{x}{2}}{2}+2\tan\frac{x}{2}\right]+c$

**Type 16**:

a. $\int\frac{a\sin x+b\cos x+c}{p\sin x+q\cos x+r}\,dx$ where a, b, c, p, q, r are any real numbers

We write

$$Numerator = A\left(denominator\right)+B\left(differ.\ of\ \ denominator\right)+C$$

where A, B and C are constant to be determined by comparing the coefficient of sin x, cos x and constant terms on both side

$$\int\frac{a\sin x+b\cos x+c}{p\sin x+q\cos x+r}\,dx=\int\frac{A\left(p\sin x+q\cos x+r\right)}{p\sin x+q\cos x+r}\,dx+\int\frac{B\left(p\cos x-q\sin x\right)}{p\sin x+q\cos x+r}\,dx+\int\frac{C}{p\sin x+q\cos x+r}\,dx$$

$$=A\int dx+B\int\frac{dt}{t}\,dx+\underbrace{\int\frac{C}{p\sin x+q\cos x+r}\,dx}_{T-15}$$

102. $\frac{3\cos x+2}{\sin x+2\cos x+3}$

103. $\frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}$

***Answers***:

102. $\frac{6}{5}x+\frac{3}{5}\log\left(\sin x+2\cos x+3\right)+-\frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+c$

103. $2x-3\tan^{-1}\left(\tan\frac{x}{2}+1\right)+c$